Numerical Solutions of Disk Source Problems

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During our study of exposing a solid surface to a pulsed laser, mathematical models including a disk source were analyzed and computed. These numerical solutions in terms of dimensionless groups are useful in heat and analogous mass transfer problems. This communication presents these solutions in graphical form for convenient use.

The energy transport equation in the solid phase is

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \qquad (1)$$

with the solid initially at $T = 0$. The disk source is located at the interface $z = 0$ over an area of πR^2. The energy input completely dissipates into the solid phase by thermal conduction.

INSTANTANEOUS DISK SOURCE

At $t = 0$, Q units of energy are released at the interface. The analytical solution is given by Carslaw and Jaeger (*1*) and involves an integration of the product of two Bessel functions. To generalize the numerical solutions for application in all systems, parameters are grouped into dimensionless quantities. The analytical solution in terms of dimensionless groups is shown in Equation (2) which, at $r = 0$, gives us Equation (3), the axial temperature distribution.

$$T_1{}^* = \frac{1}{\sqrt{\pi\theta^*}} \exp\left(-z^{*2}/\theta^*\right) \int_0^\infty \exp\left(-\theta^* \xi^2/4\right) J_0(\xi r^*) J_1(\xi) d\xi \qquad (2)$$

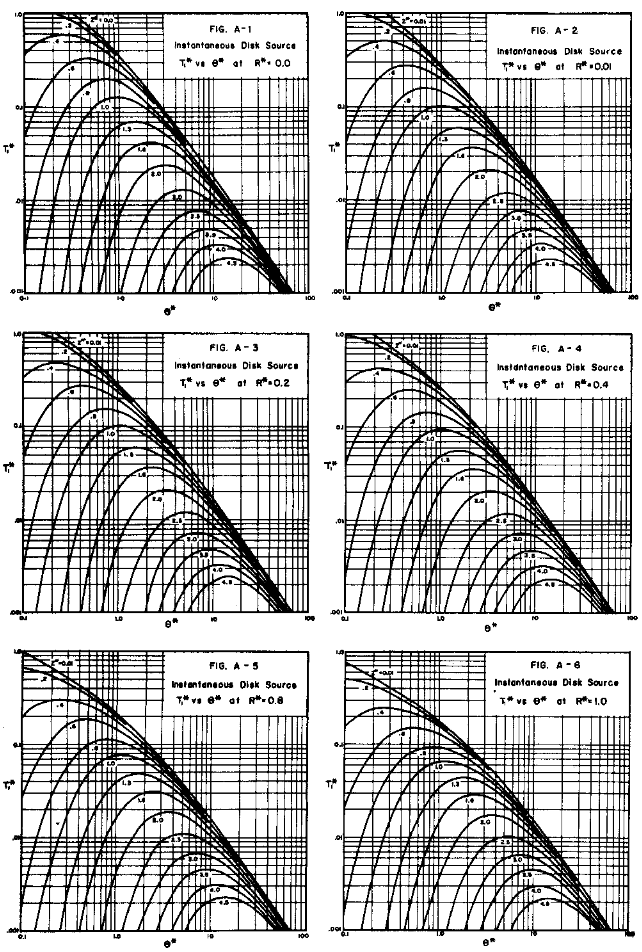

Fig. A-1 through A-6. Instantaneous disk source.

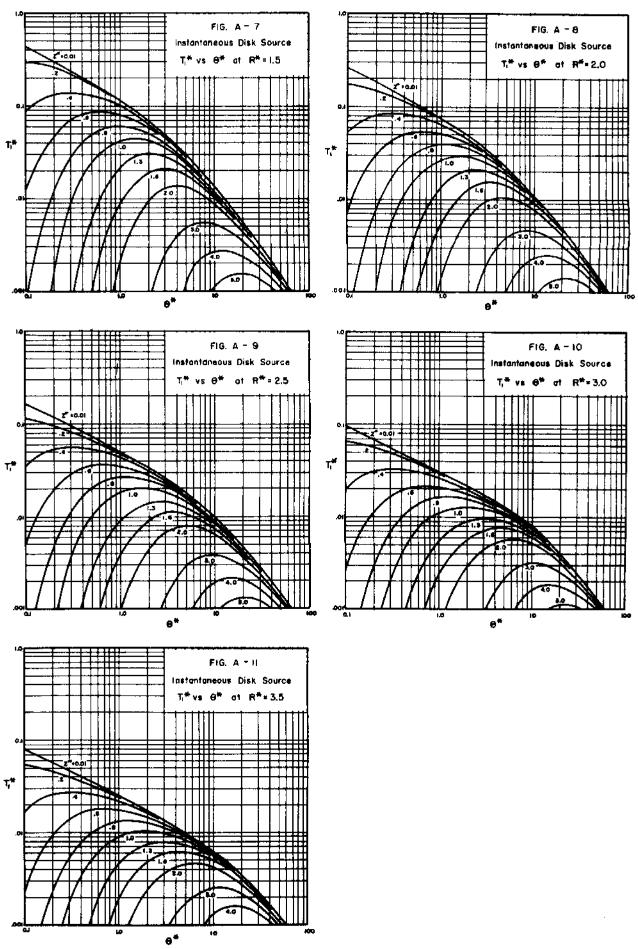

Fig. A-7 through A-11. Instantaneous disk source.

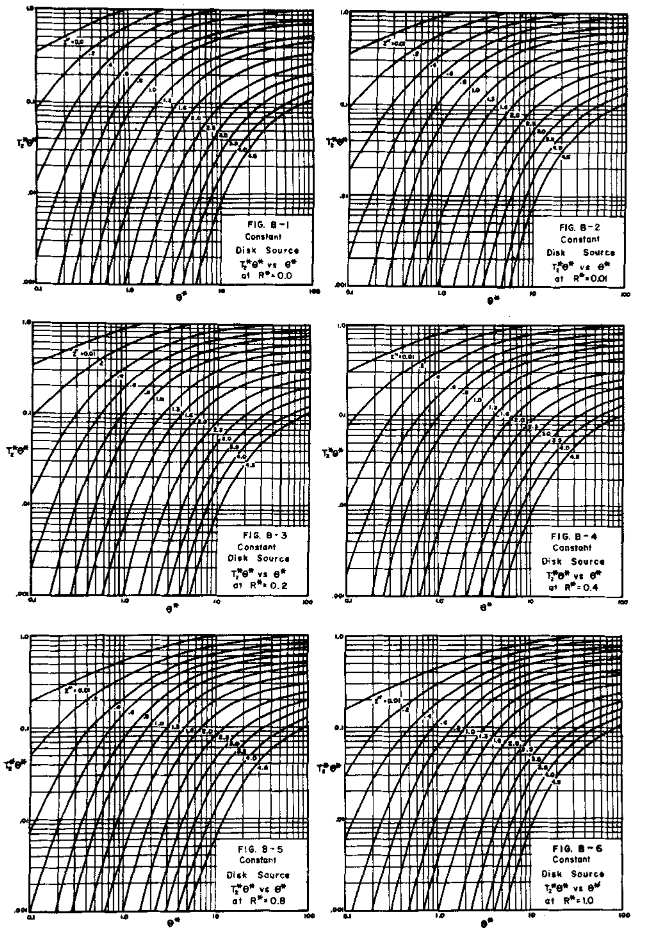

Fig. B-1 through B-6. Constant disk source.

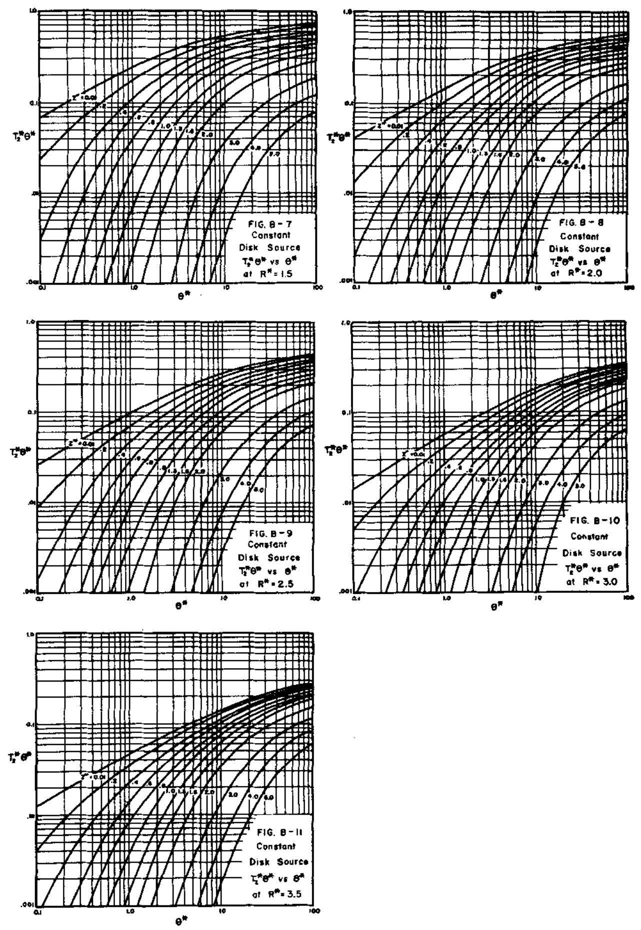

Fig. B-7 through B-11. Constant disk source.

$$T_1^* = \frac{1}{\sqrt{\pi\theta^*}} \exp\left(-z^{*2}/\theta^*\right) \left[1 - \exp\left(-1/\theta^*\right)\right] \quad \text{at} \quad r^* = 0 \qquad (3)$$

where $T_1^* = T\rho\, CR^3/2Q$
$\theta^* = 4\,\alpha\, t/R^2$
$r^* = r/R$
$z^* = z/R$
$\xi = \lambda R$

Equation (2) was solved on an IBM 1620 computer by using Simpson's Rule. Since the Bessel functions have infinite terms and the integral has infinity as its upper limit, computation errors may enter due to truncation of Bessel functions, truncation of the integral limit in addition to the conventional truncation, and round-off error associated with the chosen grid size of the independent variable. A study was made to determine the apparent optimal combination of grid size and truncation levels by observing stability and relative deviations within reasonable limits of computer time. The Bessel functions were truncated at the term which is 1% or less than the previous sum. The integral was truncated at the integral interval which is 1% or less than the integral from 0 up to that interval. The grid size chosen was $\Delta\xi = 0.05$. Convergence of computation results toward values obtained by Equation (3) proves the validity of the numerical integration method used here.

Figures in the *A*-series are results of T_1^* vs. θ^* at various levels of R^*.

CONTINUOUS DISK SOURCE

The disk source in this case is $q(t)$ energy units per unit time at time t. Since Equation (1) is linear, the superposition principle can be used to solve the continuous disk source problem.

Since an input of Q at $t = 0$ will cause a rise of temperature T_1 from time $t = 0$ to t, an instantaneous source of q at t' will cause a temperature rise from time t' to t which can be computed from Equation (2) by replacing Q by q and t by $t - t'$. Temperature T at time t, owing to a steady disk source, is the sum of all temperature rises.

$$T(t) = \int_0^t T_1(t - t')\,dt' \qquad (4)$$

If one defines a time-averaged energy rate input $\bar{q} = 1/t \int_0^t q\,dt$ and a weighting factor $w(t) = q(t)/\bar{q}$, Equation (4) can be transformed into Equation (5)

$$T_2^* = \frac{1}{\theta^*}\int_0^{\theta^*} w(\theta^* - \theta^{*\prime})\,T_1^*(\theta^* - \theta^{*\prime})\,d\theta^{*\prime} \qquad (5)$$

where $T_2^* = T\rho\, CR^3\, \pi/2\bar{q}t$. Therefore, Equation (5) can be solved by using the profile of $q(t)$ and results obtained from Equation (2), illustrated by graphs in the *A*-series here.

For the case of a constant energy source or $q(t) = \bar{q}$, $w(\theta^* - \theta)$ is unity. The numerical solution involves an integration of values on graphs of the A-series. In order to eliminate the real time in the dimensionless ordinate, $T_2^*\theta^*$ is used as the ordinate parameter for the solution of Equation (5). The solution of Equation (5) for a constant energy source is shown in graphs of series B.

From these graphs, temperature profiles for an instantaneous or continuous, constant disk source problem can be obtained. For a steady, variable disk source (that is, $w(t) \neq 1.0$) solution can be obtained by using the disk source energy profile and graphs of series A in Equation (5). For an analogous mass transfer problem involving a disk source such as the diffusion of a pollutant, concentration profiles of the diffusing component can be obtained by using the graphs presented here.

ACKNOWLEDGMENT

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NOTATION

C = heat capacity
Q = instantaneous energy source
q = continuous energy source
R = radius of disk source
r = radial distance
T = temperature
t = time
z = distance from the interface
α = thermal diffusivity
ρ = density
λ, ξ = dummy variable

LITERATURE CITED

1. Carslaw, H. S., and J. C. Jaeger, "Conduction of Heat in Solids," p. 260, Oxford Press, London (1959).

Limitations on the Generalized Differentiation Method for Obtaining Rheological Data from the Couette, Annular, and Falling Cylinder Rheometers

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In the January, 1967, issue of the *AIChE Journal*, Lescarboura, Eichstadt, and Swift (*1*) proposed a generalized method for analyzing rheological data on time independent fluids obtained from various rheometers. Implications are made that are not fully justified, and it is the purpose of this communication to discuss and clarify these matters from a theoretical standpoint.

One is led to believe from the aforementioned article that the equations obtained by the "generalized differentiation method" for the Couette, annular, and falling cylinder rheometers are valid and will lead to the *correct* flow curve, $\dot{\gamma} = f(\tau)$, for any time-independent fluid if the "flow is laminar; if there is no slip at the wall. . .; and if the rate of shear at a point depends only on the shear stress." (See the last sentence in each of the sections entitled, "The Coaxial Cylinder Rotational Rheometer," "The Concentric Annulus Rheometer," and "The Falling Cylinder Rheometer," and the remarks in the sections entitled "Discussion" and "Conclusions." It can be shown that the equations obtained for the Couette, annular, and falling cylinder rheometers by the "generalized differentiation method" of the authors are not rigorous from a theoretical